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LAMINAR THERMAL PLUME RISE IN A THERMALLY STRATIFIED ENVIRONMENT

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Abstract—The problem investigated is that of a heated laminar plume rising in a quiescent fluid with stable thermal stratification. Equations, valid for $Pr \leqslant 1$, which describe the behavior of the centerline plume velocity, temperature excess, and plume radius are developed, based on an approximate entrainment rate theory for narrow laminar flows originally proposed by Morton [2]. The theory assumes that profiles of mean vertical velocity and excess temperature are Gaussian distributions at all heights, and that the entrainment coefficient, which is the ratio of radial inflow velocity to mean centerline velocity, is a constant.

Experiments are performed with plumes generated by small disk heaters using air as the stratified fluid. The terminal height of rise of these plumes are measured for various thermal stratifi· ·tions and heat release rates.

A method is devised whereby the results for actual, finite sized heat sources can be related to the theoretical results for point sources. The experimental results are then used to determine the value of the entrainment coefficient, which is found to be nearly 1.

NOMENCLATURE

C_p, fluid specific heat;
E, entrainment coefficient;
E_a, average entrainment coefficient;
g, gravitational constant;
L, arbitrary constant (see equations (5) and (6));
Pr, Prandtl number;
Q_0, specific vertical heat flux;
r, radial coordinate;
r_s, heat source radius;
T_a, ambient temperature;
w, vertical velocity;
w_0, centerline vertical velocity;
$\bar{w}_0$, dimensionless centerline vertical velocity;
z, vertical coordinate;
$\bar{z}$, dimensionless vertical coordinate;
z_v, vertical distance between point and actual heat source;
$\bar{z}$, dimensionless vertical distance between point and actual heat source;
β, coefficient of volumetric expansion;
θ, plume temperature excess;
θ_0, plume centerline temperature excess;
$\bar{\theta}_0$, dimensionless plume centerline temperature excess;
κ, thermal diffusivity;
ν, kinematic viscosity;
ρ, fluid density;
∇, see equation (17).

INTRODUCTION

INCREASED power requirements and continued industrial growth dictate that extensive consideration be given to the problem of waste heat and effluent rejection to the environment. The behavior of the jet-like plumes which often result has received considerable attention, mainly because the often stratified structure of the receiving body places limitations on the rise height of the plume. Both forced jets and simple plumes (sources of buoyancy only) and combinations thereof exist in the environment; however, this study is concerned only with simple plumes. The results can be applied to the flow from isolated heat sources and buoyant jets with little initial momentum.

While most plumes occurring in the environment are turbulent, it is felt that further study of the behavior of laminar plumes is justified. Many laboratory models of geophysical flow behavior, in fact, operate in the laminar regime. The results are then scaled up to model the turbulent flow behavior existing in the environment. It is important, therefore, to ascertain how these laminar models behave.

Yih [1] obtained exact similarity solutions for the laminar flow above point sources of heat in a quiescent, homogeneous fluid. Morton [2] extended the entrainment rate concepts originally introduced by Morton, Taylor and Turner [3] in the analysis of turbulent plumes, to the treatment of laminar plumes. Order of magnitude analysis of the governing differential equations was used to provide the appropriate entrainment rate scale, E, which is given by

$$\int_0^{\infty} 2\pi\rho w(r,z) r \,\mathrm{d}r = \begin{cases} 2\pi E\rho\nu z & Pr \geqslant 1 \\ 2\pi E\rho\kappa z & Pr \leqslant 1 \end{cases} \tag{1}$$

with the left-hand side expressing the vertical mass flux, $w(r, z)$ is the vertical plume velocity; r and z the radial and axial coordinates. ν and κ are the kinematic viscosity and thermal diffusivity, respectively and ρ is the plume fluid density considered essentially constant except for buoyancy effects. Approximate solutions were obtained for laminar jets, plumes and wakes rising in homogeneous fluids. The numerical value of the entrainment constant was estimated by comparison with Yih's results and was found to be equal to 6. The behavior of a simple plume rising in a linearly stratified fluid was also investigated for $Pr \geqslant 1$ fluids. Equations for the behavior of centerline vertical velocity and temperature excess were obtained with the assumption of Gaussian distributions in the radial direction. Unfortunately, no estimate of the numerical value of E was given due to a lack of data in this area.

More recently, Tenner and Gebhart [4] have experimentally investigated the laminar flow of buoyant salt water jets in stratified salt solutions. They have found that under certain conditions a cellular toroidal shroud forms around the core of the jet flow. This shroud tends to isolate the jet from the surrounding fluid.

In the following, we develop approximate entrainment rate equations for the flow of a simple plume from point sources rising in a linearly stratified fluid with $Pr \leqslant 1$. The method of analysis follows that suggested by Morton [2]. A procedure for relating the point source plume to plumes above actual finite sized sources is then discussed. Finally, experiments using thermally stratified air are discussed and a value for the entrainment constant, E, is deduced from our experimental results.

THEORETICAL MODEL

Through the use of order of magnitude arguments, Morton has shown that the appropriate forms of the integral equations expressing conservation of mass, momentum, and energy in laminar plume flow are equation (1) and

$$\frac{d}{dz}\int_0^\infty r w^2(r, z)\, dr = \int_0^\infty g\beta r\theta(r, z)\, dr \tag{2}$$

and

$$\frac{d}{dz}\int_0^\infty r w(r, z)\theta(r, z)\, dr = -\frac{dT_a}{dz}\int_0^\infty r w(r, z)\, dr \tag{3}$$

where the Boussinesq approximation has been applied and the ambient fluid has constant property, linearly stratified temperature with gradient dT_a/dz. $\theta(r, z)$ is the temperature excess of the plume above the ambient temperature at that level; g and β are the acceleration of gravity and coefficient of volumetric expansion of the fluid.

For fluids with $Pr \leqslant 1$, substitution of equation (1) in (3) and integration leads to

$$\int_0^\infty r w\theta\, dr = Q_0 - \frac{E\kappa}{2}\frac{dT_a}{dz} z^2 \tag{4}$$

where Q_0 is the vertical flux of heat above the source divided by $2\pi\rho C_p$.

Next, the velocity and temperature excess profiles at any level z are assumed to be similar. As is the case in the usual Pohlhausen integral approach, any reasonably shaped profile will lead to qualitatively acceptable results. The assumption of the Gaussian distribution has proved successful in previous studies, [3] and [2], and is used here. Therefore, the velocity and temperature excess are given by

$$w(r, z) = w_0(z)\exp -\left(\frac{Lr^2}{\nu b^2}\right) \tag{5}$$

and

$$\theta(r, z) = \theta_0(z)\exp -\left(\frac{Lr^2}{\kappa b^2}\right) \tag{6}$$

where $w_0(z)$ and $\theta_0(z)$ describe the behavior of the centerline velocity and temperature excess, b is the characteristic half width of the plume, and L is an unspecified scaling constant for unit balance in the exponential terms. Substitution of (5) and (6) into (1), (2) and (4) result in three equations with unknowns $w_0(z)$, $\theta_0(z)$ and $b(z)$. Integration (with the assumption that the flow emanates from a point source of heat with zero momentum) results in

$$w_0^2(z) = \frac{1 + Pr}{PrE\kappa}\left(2g\beta Q_0 - \tfrac{1}{2}g\beta E\kappa\frac{dT_a}{dz}z^2\right) \tag{7}$$

$$\theta_0(z) = \frac{1 + Pr}{E\kappa g\beta z}\left(g\beta Q_0 - \tfrac{1}{2}g\beta E\kappa\frac{dT_a}{dz}z^2\right) \tag{8}$$

$$b^2(z) = 2L\sqrt{\left(\frac{E^3\kappa}{(1 + Pr)Pr}\right)}\left(2g\beta Q_0 - \tfrac{1}{2}g\beta E\kappa\frac{dT_a}{dz}z^2\right)^{-\frac{1}{2}} z. \tag{9}$$

Equations (7)–(9) express the variation of plume centerline vertical velocity, centerline excess temperature, and characteristic radius with height, z, above a point source of heat. These equations may be put in dimensionless form by defining dimensionless velocity, temperature excess and plume half width as

$$\bar{w}_0 = \sqrt{\left(\frac{Pr\,E}{2(1+Pr)}\frac{\kappa}{g\beta Q_0}\right)}w_0 \tag{10}$$

$$\bar{\theta}_0 = \frac{E}{1+Pr}\sqrt{\left(\frac{\kappa^5}{g\beta Q_0^3}\right)}\theta_0 \tag{11}$$

$$\bar{b} = \left[\frac{(1+Pr)Pr}{2E^3}\right]^{\frac{1}{4}}\left[\frac{\kappa}{L}\right]^{\frac{1}{2}}\sqrt{\left(\frac{g\beta Q_0}{\kappa^3}\right)}\,b \tag{12}$$

with

$$\bar{z} = \sqrt{\left(\frac{g\beta Q_0}{\kappa^3}\right)}\,z. \tag{13}$$

Then equations (7)–(9) become

$$\bar{w}_0^2 = 1 - \frac{\bar{z}^2}{\nabla^2} \tag{14}$$

$$\bar{\theta}_0 = \frac{1}{\bar{z}}\left(1 - \frac{2\bar{z}^2}{\nabla^2}\right) \tag{15}$$

$$\bar{b}^2 = \frac{\nabla\bar{z}}{(\nabla^2 - \bar{z}^2)} \tag{16}$$

where ∇ is a dimensionless parameter which characterizes the behavior of the plume given by

$$\nabla = \sqrt{\left(\frac{4g\beta Q_0^2}{E\kappa^4\dfrac{dT_a}{dz}}\right)}. \tag{17}$$

Figure 1 shows a plot of equations (14)–(16) with $\nabla = 1000$, a value of the approximate magnitude for the experiments. In this figure, $\bar{w}_0$ varies between 1 and 0 with zero slope at $\bar{z} = 1000$. θ_0 becomes small, with zero intercept at $\bar{z} = 707$. The vertical ascent of a thermal plume in a stable stratified environment stops at a definite height, owing to the vanishing of vertical velocity ($\bar{z} = 1000$) and then falls back to an equilibrium level at which there is no excess temperature, $\bar{z} = 707$. Thus, from equations (14) and (15), we can determine approximately the upper and lower extent of the plume rise as

$$\bar{z}_{max} = \nabla \tag{18a}$$

$$\bar{z}_{min} = \frac{\nabla}{\sqrt{2}}.$$

The actual cloud which would spread horizontally should lie within this band.

In the limiting case of $Pr \to 1$, the results of equations (14), (15) and (16) are identical to the approximate results obtained by Morton for $Pr \geqslant 1$. A comparison with Yih's exact results for $Pr = 1$, flow in a homogeneous fluid can also be made. For a non-stratified receiving body ($dT_a/dz = 0$) $\nabla \to \infty$ and equation (14) becomes

$$\bar{w}_0^2 = 1 \tag{19}$$

or

$$w_0^2 = \frac{2(1+Pr)}{E}\cdot\frac{g\beta Q_0}{\nu}. \tag{20}$$

In terms of our notation, Yih's expression for the centerline velocity squared becomes

$$w_0^2\Big|_{\substack{\text{exact}\\ Pr=1}} = 4\frac{g\beta Q_0}{\nu} \tag{21}$$

which implies that for $Pr = 1$, if $E = 1$, that the approximate centerline velocity result is exact. If the mass flux is calculated as

$$\dot{M} = \int_0^\infty 2\pi\rho w r\,dr \tag{22}$$

then the approximate solution results in

$$\dot{M} = 2\pi E\rho\kappa z. \tag{23}$$

The corresponding result from Yih's solution is

$$\dot{M}\Big|_{\substack{\text{exact}\\ Pr=1}} = 12\pi\rho\nu z \tag{24}$$

which implies $E = 6$ for exact agreement at $Pr = 1$.

As mentioned above, equations (18) give estimates of the upper and lower bounds of the total rise height of the plume above the point source. In order to relate this to what one would expect to observe above an actual heat source (with finite radius, r_s), the point source must be located below the actual source

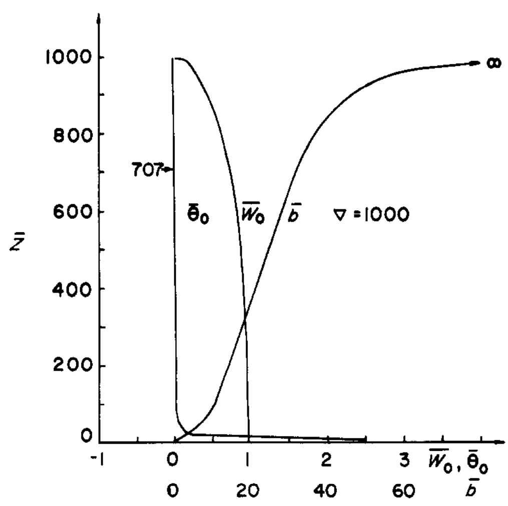

FIG. 1. Dimensionless representation of centerline velocity, temperature excess and plume half width.

as shown in Fig. 2. The observed plume height is equal to the total calculated height minus the height z_v. The distance z_v can be determined by defining the

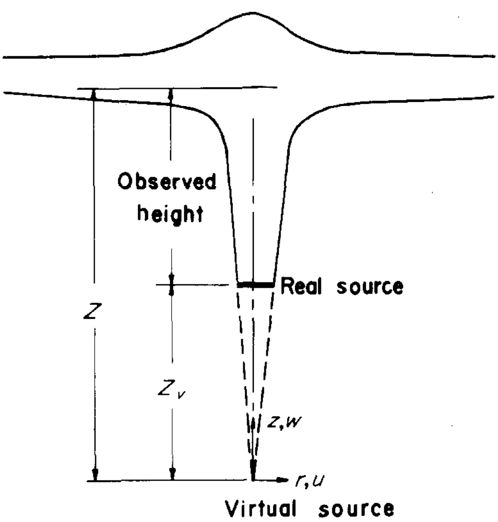

FIG. 2. The relationship between the actual heat source location and the virtual or point source.

plume radius at height z_v to be equal to the radius of the actual source, r_s, when the ratio S, of the velocity at $r = r_s$ to centerline velocity, $w_0(z_v)$, is some small fraction (for example, 01 %). We define a dimensionless scaled source radius as

$$\bar{R}^2 = \left[\frac{(1 + Pr)}{2E^3 Pr}\right]^{\frac{1}{2}} \frac{g\beta Q_0}{\kappa^3} \frac{r_s^2}{\ln(1/S)}. \tag{25}$$

Substitution into equation (5) (solve for b) and then into (9) gives

$$\bar{z}_v = \frac{\bar{R}\nabla}{(\bar{R}^2 + \nabla^2)^{\frac{1}{4}}}. \tag{26}$$

In order to estimate the terminal rise height above a finite size heat source, the result from equation (18) must be reduced by equation (26). Figure 3 shows the variation of upper and lower estimates of terminal rise height of a plume above a finite sized source for $\bar{R}^2 = 1000$. With reference to the figure, note that it can be shown that physically realistic estimates can be obtained only when $\nabla > \bar{R}$ [5].

EXPERIMENTS

Experiments were necessary in order that the value of the entrainment ratio, E, could be determined. For a given plume, with given heat release rate and ambient temperature stratification, the height at which the temperature excess vanished could be observed. The

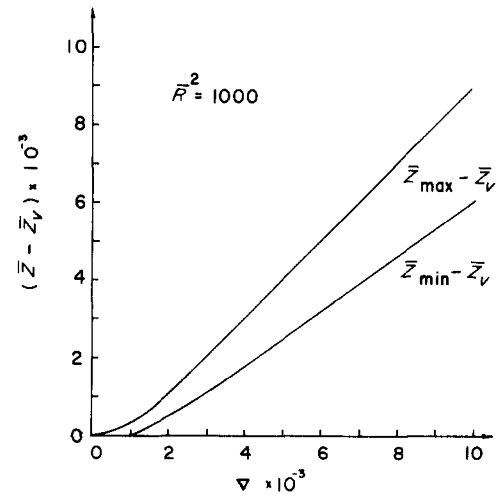

FIG. 3. Maximum and minimum plume rise heights at $\bar{R}^2 = 1000$.

value of E could then be adjusted in the dimensional forms of equations (18b) and (26) until the calculated plume rise height agreed with the height observed. Theoretically, the value of E so obtained should be a constant for all plume rise in the same fluid. To within experimental uncertainty, this was found to be the case.

The essential features of the test apparatus are shown in Fig. 4 (described in detail in reference 5).

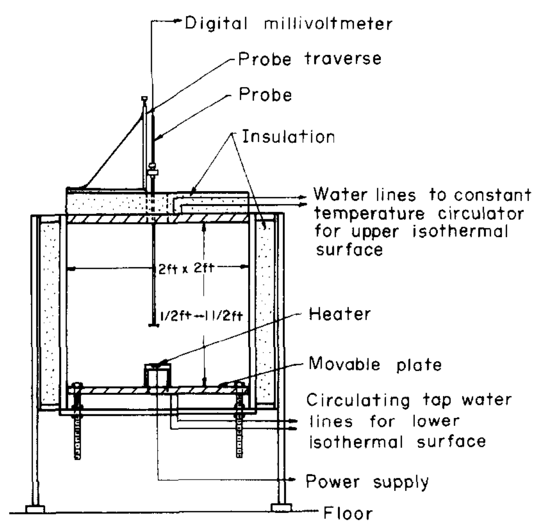

FIG. 4. Schematic of experimental apparatus.

The test region consisted of a well-insulated enclosure with aluminum inner walls. The upper and lower surfaces were hollow; water of preset temperature was circulated through them in order to maintain them isothermal and thus establish the temperature stratification in the enclosure. The temperature of each plate deviated less than 6 per cent around its set point. The aluminum side walls ensured that the enclosure boundary temperature distribution closely matched the temperature statification within. The ambient temperature gradient was selected by adjustment of the upper and lower boundary temperature and the position of the lower movable surface.

Two heat sources were used; a small solid disk heater ($\frac{3}{4}$ in. dia) and a nichrome wire grid mounted across a $\frac{1}{2}$ in. dia hole. The heat source was located on a base ≈ 3 in. above the lower isothermal surface. Heaters were supplied with d.c. power and monitored with d.c. volt and current meters.

The initial stratified temperature distribution was measured with a fixed grid of thermocouples located approximately 4 in. to the side of the expected axis of the plume. A typical temperature profile is shown in Fig. 5. Preliminary experiments with smoke indicated

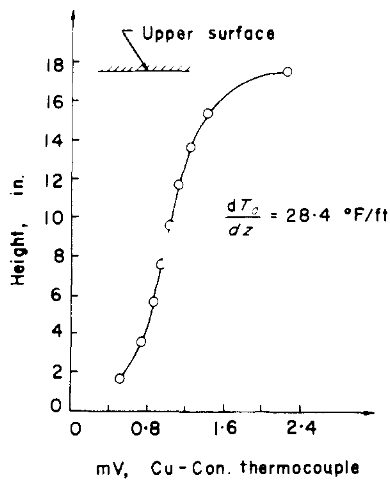

FIG. 5. Typical experimental temperature stratification.

that the central region of the enclosure was quiescent even though some fluid motion did occur in a region close to the boundary walls. This boundary motion had an unexpected beneficial effect since it tended to carry off the laterally spreading cloud which forms at the terminal rise height of the plume.

The plume temperature excess was monitored with an inverted T-shaped differential thermocouple made with 30 AWG copper/constantan wire. One junction was positioned directly above the heat source while the other was located 2 in. off the centerline axis and thus in the stratified field. All thermocouple signals were routed through a multiposition thermocouple switch to a DVM.

Once the temperature stratification was established at the desired gradient, the source was switched on at the desired heat output. Three minutes were allowed for initial start up transients to die out before any measurements were made. The T probe was then traversed and the temperature excess along the axis was recorded. The probe was traversed in both directions to eliminate hysteresis effects. During this time, observations using shadowgraph and smoke flow visualization indicated that the plume cloud remained at a fixed height. From a plot of temperature excess vs height, the observed value of $z - z_v$ could be determined.

A total of 52 data sets were obtained, 25 for the solid disk heater, and 27 for the wire grid heater. Ten different temperature stratifications ranging from 20°F/ft to 29·5°/ft were used and the heater outputs were varied from 0·293W to 1·85W. Observed rise heights ranged from approximately 5 in. to 13 in.

For each data set, the value of E which produced a plume rise height, $z - z_v$, equal to the observed height could be selected by trial and error once the ratio of velocities S was selected. The selection of the value of S is arbitrary except that it should be small if it is to adequately match the heater size to the plume size. Values of S ranging from $S = 0{\cdot}01$ to $S = 0{\cdot}3$ were investigated for each heat source geometry. The average value of the values and rms deviation of E so determined are tabulated in Table 1 for five values of S. The difference in average E_a at fixed S between heater sources is attributed to the difference in heater geometry used. With the disk heater, the plume fluid had to approach the heat source laterally and then turn upward, while with the grid heater, the plume flow was vertical upward through the heater. Thus, for equal heater outputs, the temperature excess immediately above the grid heater was probably greater than that above the disk heater resulting in greater plume rise. Reference to equation (18b) indicates that this would, in general, require a smaller value of E.

A comparison between observed and predicted rise height for $E_a = 1{\cdot}0$, $S = 0{\cdot}01$ is shown in Fig. 6 for our data. From this we see that the value of E suggested by a comparison with Yih's exact velocity profile for homogeneous plume rise is adequate over the range of source strengths and stratification parameters tested.

CONCLUSION

Appropriate equations expressing the behavior of centerline velocity, temperature excess, and characteristic half width have been developed for simple

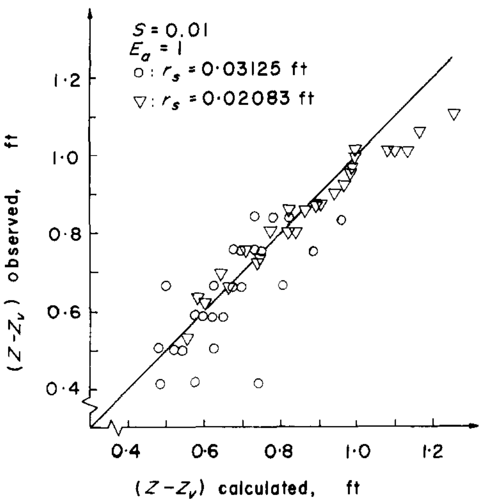

FIG. 6. Comparison of observed and theoretical plume rise heights for $E_a = 1$.

Table 1. The average values of entrainment coefficient and its rms deviation about the average for different velocity ratios

	S	0·01	0·05	0·1	0·2	0·3
Disk heater	E_a	1·023	2·224	3·568	5·400	7·336
	rms %	22·57	18·40	16·05	23·69	32·65
Grid heater	E_a	0·924	1·420	1·847	2·643	3·533
	rms %	7·620	15·86	25·96	37·99	46·33

plumes rising in linearly stratified fluids wlth $Pr \leqslant 1$. The equations appear to adequately describe the general behavior of these plumes in that in the limiting case of zero stratification they have the same functional form as exact solutions.

The numerical value of the entrainment ratio, $E \simeq 1$, suggested by a comparison with Yih's exact solution, is supported by our experimental results using air flow. A further comparison between equations (20) and (21) suggests that E may vary with Prandtl number as

$$E \propto \frac{1 + Pr}{2}. \tag{27}$$

Finally, our experience indicates that experiments with density stratified fluids can conveniently be conducted using air thermal stratifications. Experiments of this kind have usually been conducted using salt solution stratifications ($Pr \simeq 6{\cdot}25$). The disadvantage is that once disturbed the stratification must be re-established by refilling the apparatus. With our experimental configuration, however, the true steady state density distribution is stratified as established by the temperature of the upper and lower boundaries of the enclosure. If the fluid is disturbed, it rapidly ($\approx$ 15 min) will return to the stratified condition, ready for further testing.

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MONTEE D'UN PANACHE LAMINAIRE DANS UN ENVIRONNEMENT THERMIQUEMENT STRATIFIE

Résumé—Le problème étudié est celui d'un panache laminaire chaud qui s'élève dans un fluide au repos et stratifié thermiquement de façon stable. On établit des équations, valides pour $Pr < 1$, qui décrivent les variations de la vitesse sur l'axe, de l'excès de température, du rayon du panache. Elles sont basées sur une théorie approchée de l'entrainement pour des écoulements laminaires étroits. Cette théorie proposée par Morton (2) suppose que les profils de la vitesse moyenne verticale et de l'excès de température sont des distributions gaussiennes à toute altitude et que le coefficient d'entrainement, rapport de la vitesse radiale à la vitesse moyenne sur l'axe, est constant.

On réalise des expériences avec des panaches générés par des petits chauffoirs en forme de disque, l'air étant le fluide stratifié. La hauteur extrême de montée de ces panaches est mesurée pour différentes stratifications et différentes quantités de chaleur évacuées.

On développe une méthode qui relie les résultats obtenus avec des sources de taille finie aux résultats théoriques relatifs aux sources ponctuelles. Les résultats expérimentaux sont utilisés pour déterminer la valeur du coefficient d'entrainement qui est trouvé proche de l'unité.

LAMINARE THERMISCHE AUFTRIEBSSÄULEN IN THERMISCH GESCHICHTETER UMGEBUNG

Zusammenfassung—Es wird eine erhitzte laminare Auftriebssäule in einem ruhenden Fluid mit stabiler thermischer Schichtung untersucht. Es wurden Gleichungen entwickelt gültig für $Pr < 1$, welche das Verhalten der Zentralgeschwindigkeit in der Fluidsäule der Übertemperatur und des Säulenradius beschreiben. Sie beruhen auf einer angenäherten Theorie über Mitreissraten, für enge laminare Strömungen wie sie ursprünglich von Morton [2] vorgeschlagen wurde. Die Theorie nimmt an, dass die Profile der mittleren Vertikalgeschwindigkeit und der Übertemperatur auf allen Höhen Gaussverteilungen aufweisen und dass der Mitreisskoeffizient, als Verhältnis von radialer Zuflussgeschwindigkeit zu mittlerer Zentralgeschwindigkeit konstant ist. Experimente wurden ausgeführt, wobei die Säulen durch kleine Scheibenheizungen angeregt wurden, mit Luft als dem geschichteten Fluid. Die Gipfelhöhen dieser Säulen wurden für mehrere thermische Schichtungen und Wärmeströme gemessen. Es wurde eine Methode ausgearbeitet, wonach Ergebnisse an wirklichen, endlich grossen Wärmequellen auf theoretische Ergebnisse für punktförmige Quellen bezogen werden können. Die experimentellen Ergebnisse werden dann verwendet, um den Wert des Mitreisskoeffizienten zu bestimmen, welcher ungefähr 1 beträgt.

ЛАМИНАРНЫЙ ПОДЪЕМ НАГРЕТОГО ОБЪЕМА В СРЕДЕ С ТЕМПЕРАТУРНОЙ СТРАТИФИКАЦИЕЙ

Аннотация—Исследовался случай, когда нагретый объем ламинарно поднимается в неподвижной среде с устойчивой температурной стратификацией на основе приближенной теории скорости уноса для узких ламинарных потоков, первоначально предложенной Мортоном [2]. Выведены уравнения, которые сраведливы для $Pr < 1$ и с помощью которых описываются изменения скорости объема в центре, приращение температуры и радиус объема. Предполагается, что на любой высоте профили средней вертикальной скорости и избыточной температуры являются гауссовыми распределениями и что коэффициент уноса, представляющий собой отношение радиальной скорости натекания к средней скорости в центре, является величиной постоянной.

Эксперименты проводились с объемами, образуемыми небольшими дисковыми нагревателями. В качестве среды использовался воздух. Для различных температурных стратификаций и скоростей отдачи тепла измерялась конечная высота подъема объема.

Разработан метод, с помощью которого результаты для фактических источников тепла конечных размеров можно сопоставить с теоретическими данными для точечных источников. Кроме того, с помощью экспериментальных данных определялся коэффициент уноса, который, как найдено, равен приблизительно 1.